Announcements

1) HW2 due Thursday!

2) MATLAB training sessions, 2/20
Kochoff C
10 – 12 (basic)
1 – 3:30 (advanced)

The QR Decomposition

Remember Gram Schmidt!

Given $\{x_1, x_2, \ldots, x_n\}$

linearly independent and

nonzero. Gram-Schmidt

produces $\{y_1, \ldots, y_n\}$

orthogonal with

$\text{span}(\{x_1, \ldots, x_n\}) = \text{span}(\{y_1, \ldots, y_n\})$

$$y_1 = x_1 / \|x_1\|_2$$

$$y_2 = \frac{x_2 - (y_1^* x_2) y_1}{\|x_2 - (y_1^* x_2) y_1\|_2}$$

$$y_3 = \frac{x_3 - (y_2^* x_3) y_2 - (y_1^* x_3) y_1}{\|x_3 - (y_2^* x_3) y_2 - (y_1^* x_3) y_1\|_2}$$

$$\vdots$$

$$y_n = \frac{x_n - \sum_{i=1}^{n-1} (y_i^* x_n) y_i}{\|x_n - \sum_{i=1}^{n-1} (y_i^* x_n) y_i\|_2}$$

This procedure produces an orthonormal set and is the basis for the existence of orthogonal projections in finite dimensions:

Let V be finite dimensional over $\mathbb{R}$ or $\mathbb{C}$, let W be a subspace of V.

Let $\{x_1, \dots, x_n\}$ be a basis for W. Let $\{y_1, \dots, y_n\}$ be the vectors from Gram-Schmidt.

Define the orthogonal projection $P: V \to W$ by

$$\boxed{P(x) = \sum_{i=1}^{n} (y_i^* x) y_i}$$

$\forall x \in V$.

The QR decomposition is the Gram-Schmidt algorithm in matrix form.

Example 1: $A = \begin{bmatrix} 1 & i \\ 2 & 6 \end{bmatrix}$

The vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} i \\ 6 \end{bmatrix}$ are linearly independent. Use Gram-Schmidt, but as matrices!

Write $x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $x_2 = \begin{bmatrix} i \\ 6 \end{bmatrix}$

Step 1: Normalize x_1

$$\begin{bmatrix} 1 & i \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{5} & i \\ 2/\sqrt{5} & 6 \end{bmatrix}$$

Let $q_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

Step 2: Orthogonalize the columns (without changing q_1).

$$\begin{bmatrix} 1/\sqrt{5} & i \\ 2/\sqrt{5} & 6 \end{bmatrix} \begin{bmatrix} 1 & -q_1^* x_2 \\ 0 & 1 \end{bmatrix}$$

$$q_1^* x_2 = \begin{bmatrix} 1/\sqrt{5} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} i \\ 6 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}}(i+12)$$

$$\begin{bmatrix} 1/\sqrt{5} & i \\ 2/\sqrt{5} & 6 \end{bmatrix} \begin{bmatrix} 1 & \frac{-1}{\sqrt{5}}(i+12) \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{5} & \frac{-1}{5}(i+12)+i \\ 2/\sqrt{5} & -\frac{2}{5}(i+12)+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{5} & -\frac{12}{5}+\frac{4i}{5} \\ 2/\sqrt{5} & \frac{6}{5}-\frac{2i}{5} \end{bmatrix} \checkmark$$

Step 3: Normalize the second column.

$$\left\| \begin{bmatrix} -\frac{12}{5} + \frac{4i}{5} \\ \frac{6}{5} - \frac{2i}{5} \end{bmatrix} \right\|_2$$

$$\frac{2}{5} \left\| \begin{bmatrix} -6+2i \\ 3-i \end{bmatrix} \right\|_2$$

$$= \frac{2}{5}\sqrt{40 + 10}$$

$$= \frac{2\sqrt{50}}{5}$$

$$= 2\sqrt{2}$$

$$\begin{bmatrix} 1/\sqrt{5} & -\frac{12}{5} + \frac{4i}{5} \\ 2/\sqrt{5} & \frac{6}{5} - \frac{2i}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{5} & \frac{1}{5\sqrt{2}}(-6+2i) \\ 2/\sqrt{5} & \frac{1}{5\sqrt{2}}(3-i) \end{bmatrix}$$

$$= Q$$

Then Q is a unitary matrix!

Moreover, Q was obtained from A by multiplying A on the right by

$$T = \begin{bmatrix} 1/\sqrt{5} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{\sqrt{5}}(i+12) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2\sqrt{2} \end{bmatrix}$$

$=$ some upper triangular matrix

(invertible!)

Definition: (reduced QR)

Let $A \in \mathbb{C}^{m \times n}$. The **reduced QR factorization** of A is the decomposition

$$\boxed{A = \hat{Q}\hat{R}}$$

with $\hat{Q} \in \mathbb{C}^{m \times n}$ with orthonormal columns and $\hat{R} \in \mathbb{C}^{n \times n}$ upper triangular.

Just like SVD, $\exists$

full QR:

$$A = QR$$

with $Q \in \mathbb{C}^{m \times m}$ unitary

and $R \in \mathbb{C}^{m \times n}$ "upper triangular"!

Theorem: (Existence of QR decomposition)

Let $A \in \mathbb{C}^{m \times n}$. Then A has both a full and reduced QR decomposition. The decomposition $A = \hat{Q}\hat{R}$ is unique up to signs.

proof: Gram-Schmidt!

$\square$

Matlab Calling Command

$[Q,R] = qr(A,0)$ (reduced)

$[Q,R] = qr(A)$ (full)

Example 2: $A = \begin{bmatrix} 1 & -1 \\ 3+i & 2 \\ 0 & 6 \end{bmatrix}$

Using Matlab,

$$\hat{Q} = \begin{bmatrix} -.3015 & .2348 - .0294i \\ -.9045 - .3015i & -.0734 - .0147i \\ 0 & -.9687 \end{bmatrix}$$

$$\hat{R} = \begin{bmatrix} -3.3166 & -1.5076 + .6030i \\ 0 & -6.1938 \end{bmatrix}$$

$$Q = \begin{bmatrix} -.3015 & .2348 - .0294i & .8964 - .2227i \\ -.9045 - .3015i & -.0734 - .0147i & -.2912 - .0228i \\ 0 & -.9687 & .2465 - .0295i \end{bmatrix}$$

$$R = \begin{bmatrix} -3.3166 & -1.5076 + .6030i \\ 0 & -6.1938 \\ 0 & 0 \end{bmatrix}$$

Notice how Q and R are obtained from $\hat{Q}$ and $\hat{R}$!

Algorithm for QR Decomposition

Matlab!

A a matrix, in Matlab,

$A(:, j)$ returns the j^{th} column of A.

$A(i,:)$ returns the i^{th} row.